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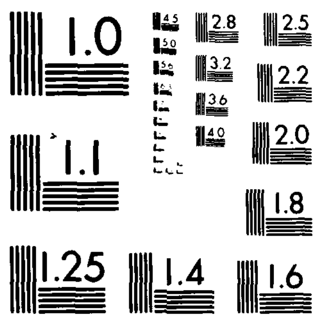

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THE UNIVERSITY OF IOWA

DEPARTMENT OF STATISTICS

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ON APPROXIMATION OF THE LEVEL PROBABILITIES AND ASSOCIATED DISTRIBUTIONS IN ORDER RESTRICTED INFERENCE

TECHNICAL REPORT NO. 81

ON APPROXIMATION OF THE LEVEL PROBABILITIES AND ASSOCIATED DISTRIBUTIONS IN ORDER RESTRICTED INFERENCE[1]

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Tim Robertson and F. T. Wright

SUMMARY

The use of much of the distribution theory developed for order restricted inference has been limited by the lack of computation algorithms for the "level probabilities" encountered in that theory. An approximation for these level probabilities, which accounts for the pattern of large and small "weights," is developed. This approximation and the equal weights approximation are examined in several different ways including the use of randomly generated weight sets. Both approximations appear to be reasonable for weight sets having a moderate amount of variability. The quality of the equal weights approximation, as a function of the amount of variability in the weights, deteriorates more quickly for certain patterns of large and small weights than for others. Thus, the approximation based upon the pattern of large and small weights is a significant improvement over the equal weights approximation. Finally, Siskind's (1976) approximation, which can be applied if the number of parameters is not too large, is discussed.

Key words and phrases: Order restricted inference, level probabilities, chi-bar-square distribution.

1. Introduction. The chi-bar-square ($\bar{\chi}^2$) and E-bar-square ($\bar{E}^2$) distributions are fundamental to the theory of order restricted hypothesis tests. For a comprehensive treatment of the early work in order restricted inference the reader may consult Barlow, Bartholomew, Bremner and Brunk (1972). These distributions have tail probabilities which are linear combinations of the tail probabilities of standard distributions and they depend upon the order restriction through the coefficients in these linear combinations. The values of these coefficients are the probabilities that the order restricted maximum likelihood estimates of normal means assume specified numbers of distinct values, which are called levels. These probabilities are computed under the hypothesis that the population means are equal. The maximum likelihood estimates are based upon independent samples from each of the populations and depend upon the vector of relative precisions of the sample means as estimates of the corresponding population means. The precisions will be referred to as weights. The use of these tests has been limited by the fact that these level probabilities can be virtually impossible to compute if the weights are not all equal. In this paper we describe a technique for approximating these level probabilities for a linear order restriction and for unequal precisions. This approximation is based upon an idea of Chase (1974) and uses the pattern of large and small relative precisions. It seems to be particularly good when the relative precisions have two distinct values. However, it seems to provide a satisfactory approximation as long as the variation in the relative precisions is not too large. For example, our study of the case of five means indicates that if the ratio of the largest relative precision to the smallest relative precision is no more than 4.7 then this approximation provides

an adequate approximation for the tail probabilities of the $\bar{\chi}^2$ distribution. This is probably due to the lack of sensitivity of the level probabilities to changes in the weights. This "robustness" was noticed by both Grove (1980) and Siskind (1976). Siskind conjectured that, because of this robustness, the equal weights case should provide reasonable approximations except in extreme cases.

In Section 2 we provide some background on the computation of these level probabilities and report upon our investigation of Siskind's conjecture about the adequacy of the equal weights approximation. It appears that the quality of the equal weights approximation is related to the pattern of large and small weights and in Section 3 we explore another approximation which is based upon such patterns. The equal weights case is used as a benchmark for judging this new approximation. In Section 4 we briefly discuss Siskind's (1976) approximation for the level probabilities.

2. Preliminaries and the equal weights approximation. Suppose we have independent random samples from each of k normal populations having means $\mu_1,\mu_2,\cdots,\mu_k$ and assume that the corresponding sample means and sample sizes are denoted by $\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_k$ and $n_1,n_2,\cdots,n_k$. Let $P_w(\ell,k)$ denote the probability that the maximum likelihood estimates of $\mu_1,\mu_2,\cdots,\mu_k$ subject to the linear order restriction $H:\mu_1 \le \mu_2 \le \cdots \le \mu_k$ have exactly ℓ distinct levels. These level probabilities are computed under the assumption that $\mu_1 = \mu_2 = \cdots = \mu_k$ and they depend upon the vector $w = (w_1,w_2,\cdots,w_k)$ of relative precisions of the sample means, $\bar{x}_i$, as estimates of the corresponding population means, μ_i; $i = 1,2,\cdots,k$. We are assuming that either the population variances, σ_i^2; $i = 1,2,\cdots,k$ are known or if unknown they are assumed equal. The value of w_i is n_i/σ_i^2

in the former case and this value is n_i in the latter.

For $k \leq 5$ and for arbitrary weights, formulas for $P_w(\ell,k)$ are given on pages 140-142 of Barlow et al. (1972). There are typographical errors in the formulas for $P_w(3,5)$ and $P_w(1,4)$. The formula for $P_w(5,5)$ involves an integral which may be evaluated numerically or approximated using methods described in Barlow et al. (1972). Cran (1981) gives a computer algorithm for the computation of $P_w(\ell,k)$ for $k \leq 4$ and for their approximation for $k = 5,6$. For $k = 5$ he uses an approximation due to Plackett (1954) and for $k = 6$ he uses an approximation due to Childs (1967).

For the equal weights case, that is, $w_1 = w_2 = \cdots = w_k$, a recursion formula for computing $P_w(\ell,k)$ is given in Corollary B on page 145 of Barlow et al. (1972). This formula is a quite satisfactory computation algorithm for this important special case. We denote the equal weights level probabilities by $P(\ell,k)$. The generating function of the sequence $\{P(\ell,k)\}_{\ell=1}^{k}$ is given by

$$\Phi_k(s) = \binom{s+k-1}{k} = \frac{s(s+1)\cdots(s+k-1)}{k!}.$$

Grove (1980) and Siskind (1976) both claimed that the $P_w(\ell,k)$'s are not very sensitive to changes in the weights and that the equal weights case should provide reasonable approximations except for extreme cases. Siskind felt that the pattern of large and small weights was important and that approximation is difficult for a U-shaped configuration of weights, that is, for weight sets with w_1 and w_k relatively large and the other w_i small.

For the most part, we will judge approximations as they apply to

Bartholomew's $\bar{\chi}^2$ distribution. The tail probabilities of this distribution are given by

$$\mathrm{pr}_w[\bar{\chi}_k^2 \geq t] = \Sigma_{\ell=1}^k P_w(\ell,k)\mathrm{pr}[\chi_{\ell-1}^2 \geq t]$$

where $\chi_{\ell-1}^2$ denotes a standard chi-square variable having $\ell-1$ degrees freedom $(\chi_0^2 \equiv 0)$. Distributions, which have tail probabilities which are linear combinations of the tail probabilities of standard distributions arise in several restricted testing problems. In addition to those discussed in Barlow et al. (1972), see Robertson and Wegman (1978), Robertson (1978), Grove (1980), Robertson and Wright (1981), Robertson and Warrack (1981), and Dykstra and Robertson (1981a,b).

Robertson and Wright (1982) derive upper and lower bounds for the $\bar{\chi}^2$ and $\bar{E}^2$ distributions and prove that, in general, these bounds cannot be improved upon. The lower bound is approached using $\cup$-shaped weight sets of the form $(a,\epsilon,\epsilon,\cdots,\epsilon,b)$ where $a,b > 0$ and $\epsilon \to 0$. The upper bound is approached using $\cap$-shaped weight sets of the form $(\epsilon^H,\epsilon^{H-1},\cdots,\epsilon,1,\epsilon,\cdots,\epsilon^{H-1},\epsilon^H)$ and letting $\epsilon \to 0$. This is for odd k where $k = 2H+1$. The obvious adjustments are made for even k.

These bounds can differ substantially. For example, for $k = 5$ the 5 percent, equal weights critical value is 5.049. The upper and lower bounds for $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$ are .013 and .096.

What is the rate of convergence of $\mathrm{pr}_w[\bar{\chi}_5^2 \geq t]$ *to the bounds?*

We computed $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$ for various weight sets. Since k is 5 we can compute, using a numerical integration on (3.18) in Barlow et al.

(1972), an exact value for $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$. If the equal weights approximation is good then this should be close to .05. For each weight set we also computed the ratio, R, of the largest weight to the smallest weight in order to gain some feel for the likelihood that the weight set might be encountered in practice. For U-shaped configurations we let $w = (1,\epsilon,\epsilon,\epsilon,1)$ with ϵ chosen so that $R = \epsilon^{-1}$ ranged between 1 and 30. The results are given in Table 1. The probabilities, $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$ decreased from .05 at $\epsilon = 1$ to .021 at $\epsilon = .0333$. We see that the lower bound, .013, is approached rather slowly.

For ∩-shaped configurations we let $w = (\epsilon,\sqrt{\epsilon},1,\sqrt{\epsilon},\epsilon)$ with ϵ again chosen so that $R = \epsilon^{-1}$ ranged between 1 and 30. These results are also given in Table 1. These probabilities increased from .05 to .075. Thus, it seems that while the range of probabilities is considerable it is not as extreme as the bounds suggest, at least for weight sets usually encountered in practice.

Are U-shaped and ∩-shaped the only configurations of weights for which the equal weights case gives a poor approximation?

We carried out a "random search" for weight configurations for which $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$ is not close to .05. Five uniform [0,1] random numbers, $U_1,U_2,\cdots,U_5$, were generated and, in order to make the weight sets comparable, we set $w_i = U_i / \sum_{j=1}^{5} U_j$; $i = 1,2,\cdots,5$. We repeated this process 10,000 times and for each weight set we computed $\mathrm{pr}_w[\bar{\chi}_5^2 \geq 5.049]$ and the ratio of the largest weight to the smallest weight in the weight set. The 10,000 ratios ranged from 1.1 to something larger than 10,000. Recall that, because of the bounds, these probabilities must fall between .013 and .096.

In Table 2 we have reported a frequency distribution for these 10,000 probabilities. All 10,000 probabilities fell between .0219 and .0803.

ϵ	R	$w = (1,\epsilon,\epsilon,\epsilon,1)$ $pr_w[\bar{\chi}_5^2 \geq 5,049]$	$w = (\epsilon,\sqrt{\epsilon},1,\sqrt{\epsilon},\epsilon)$ $pr_w[\bar{\chi}_5^2 \geq 5.049]$
1.0000	1	.0500	.0500
.5000	2	.0420	.0557
.3333	3	.0375	.0590
.2500	4	.0346	.0612
.2000	5	.0324	.0629
.1000	10	.0267	.0679
.0667	15	.0240	.0705
.0500	20	.0224	.0723
.0400	25	.0213	.0736
.0333	30	.0205	.0746

Table 1. $pr_w[\bar{\chi}_5^2 \geq 5.049]$ for various weight configurations.

Interval	Frequency	Minimum Ratio	Maximum Ratio	Interval	Frequency	Minimum Ratio	Maximum Ratio
[.012,.016]	0			[.056,.060]	1116	2.2	610.2
[.016,.020]	0			[.060,.064]	670	3.4	∞
[.020,.024]	4	20.8	∞	[.064,.068]	357	5.0	4777.9
[.024,.028]	22	18.3	571.2	[.068,.072]	145	6.9	9543.2
[.028,.032]	73	8.0	2450.7	[.072,.076]	54	14.9	2075.1
[.032,.036]	284	5.2	∞	[.076,.080]	17	34.5	1622.1
[.036,.040]	719	3.4	∞	[.080,.084]	2	812.0	1597.7
[.040,.044]	1319	2.0	4052.0	[.084,.088]	0		
[.044,.048]	1781	1.4	3531.9	[.088,.092]	0		
[.048,.052]	1876	1.1	2996.0	[.092,.096]	0		
[.052,.056]	1561	1.4	1648.8				

Table 2. Frequency distribution for $pr_w[\bar{\chi}_5^2 \geq 5.049]$ with 10,000 randomly generated weight sets.

The number of these probabilities which fell between .040 and .060 was 7353. We would feel that for these weight configurations the equal weights case provided an adequate approximation.

Large ratios are not necessarily associated with poor approximations. However, every ratio which is associated with a probability outside of [.048,.052] is at least 1.4, indicating that if the ratio of the largest weight to the smallest weight is less than 1.4 the equal weights case should provide a very good approximation. Every ratio which is associated with a probability outside of .04 to .06 is at least 3.4 indicating that if the ratio of the largest weight to the smallest weight is less than 3.4 then the equal weights case should provide an adequate approximation.

In another study, the weights were randomly generated in such a way that the ratio of the largest weight to the smallest weight was constrained so that it will not exceed a particular value. For example, we randomly generated the values of U_i (recall that $w_i = U_i/\Sigma U_i$) in the interval [1/2,3/2]. Thus, the ratio will not exceed 3. For 1000 such randomly generated weight sets all 1000 values of $pr_w[\bar{\chi}_5^2 \geq 5.049]$ fell in the interval [.0410,.0592]. Results for other constraints are given in Table 3.

The quality of the equal weights approximation is related to both the amount of variability in the weights and to the pattern of large and small weights. In fact, it may be more directly related to this pattern than to the amount of variability. If we interpret U-shaped, very broadly, to mean any weight pattern with relatively large weights at 1 and k but at least one relatively small weight between 1 and k then all of the weight configurations for which $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is very small have a U-shaped configuration. A selection of 10 of these weights is given in Table 4.

[a,b]	All 1000 values of $pr_w[\bar{\chi}_5^2 \geq 5.049]$ in:
[1/9,10/9]	[.0328,.0670]
[1/4,5/4]	[.0372,.0627]
[1/3,4/3]	[.0389,.0613]
[1/2,3/2]	[.0410,.0592]
[1,2]	[.0443,.0559]
[2,3]	[.0466,.0535]

Table 3. Limits on $pr_w[\bar{\chi}_5^2 \geq 5.049]$ when weights are generated to control the ratio of the largest to the smallest weight.

w	R	$pr_w[\bar{\chi}_5^2 \geq 5.049]$
(.4333,.0049,.0912,.0451,.4256)	88.4	.0261
(.4887,.0941,.0562,.0069,.3541)	70.8	.0273
(.3288,.0126,.3777,.0024,.2785)	157.4	.0312
(.3521,.0757,.0059,.1899,.3764)	63.8	.0314
(.3168,.1112,.0078,.0459,.5183)	66.4	.0274
(.5392,.0059,.0664,.0607,.3278)	91.4	.0263
(.4124,.0078,.1156,.0420,.4222)	54.1	.0274
(.5459,.0012,.2413,.0334,.1782)	454.9	.0316
(.5473,.0910,.0240,.0366,.3012)	22.8	.0279
(.4[illegible],.0006,.1629,.0984,.2789)	765.3	.0311

Table 4. Weight configurations for which $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is small.

In Robertson and Wright (1982a), it was shown that as $\epsilon \to 0$, the distributions corresponding to the weight sets $(1, \epsilon, \epsilon^2, \cdots, \epsilon^{k-1})$ and $(\epsilon^{k-1}, \epsilon^{k-2}, \cdots, 1)$ also approach the upper bound for such $\bar{\chi}^2$ distributions. So we interpret $\cap$-shaped broadly to mean any pattern with small weights in both extremes or several adjacent small weights at either extreme. All of the weight configurations for which $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is very large have such a $\cap$-shape. A selection of 10 of these weight configurations is given in Table 5.

In Table 6 we have listed a selection of 10 weight sets for which R is large, but $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is close to .05. In these cases there was a small weight at one or both extremes and another at a nonadjacent interior position.

In summary, the equal weights case appears to give fair approximations except for weight sets having a U-shaped or a $\cap$-shaped (interpreted broadly) pattern of large and small weights. Regardless of this pattern, if the ratio of the largest weight to the smallest weight is less than 1.4, the equal weights case gives a very satisfactory approximation and if this ratio is less than 3.4 it appears to give an adequate approximation. If the pattern of large and small weights is not either U-shaped or $\cap$-shaped then the equal weights case can give a satisfactory approximation for weight sets having ratios much larger than 3.4.

3. <u>An approximation based upon the pattern of large and small weights.</u>

Chase (1974) found a good approximation for an important special case. He was thinking about an experimental situation in which a researcher was

w	R	$pr_w[\bar{\chi}_5^2 \geq 5.049]$
(.0005,.0382,.4538,.3664,.1410)	907.6	.0789
(.0128,.1719,.4122,.3340,.0690)	32.2	.0729
(.3101,.4265,.2366,.0239,.0029)	147.1	.0721
(.0747,.1946,.3779,.3523,.0005)	755.8	.0730
(.0373,.1172,.5135,.3316,.0003)	1711.7	.0770
(.0065,.0902,.2685,.5091,.1257)	78.3	.0727
(.0411,.2618,.6850,.0117,.0004)	1712.5	.0827
(.0011,.0370,.3114,.4106,.2399)	373.3	.0757
(.2622,.1880,.5439,.0055,.0004)	1359.7	.0721
(.0682,.2726,.3317,.3271,.0004)	829.5	.0740

Table 5. Weight configurations for which $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is large.

w	R	$pr_w[\bar{\chi}_5^2 \geq 5.049]$
(.0225,.2335,.4097,.0016,.3328)	273.1	.0491
(.1118,.1941,.0032,.6453,.0457)	201.7	.0494
(.0024,.3551,.0137,.1820,.4467)	186.1	.0492
(.0314,.5413,.0002,.2894,.1377)	2706.5	.0492
(.1276,.0017,.4400,.3283,.1023)	258.8	.0489
(.1232,.0030,.2307,.5197,.1234)	173.2	.0481
(.0636,.3279,.0030,.5030,.1025)	167.7	.0510
(.2966,.0783,.0806,.5392,.0052)	103.7	.0503
(.5517,.0142,.2524,.1777,.0041)	134.6	.0515
(.0005,.3842,.0349,.1678.,4126)	825.2	.0516

Table 6. Weight configurations for which the ratio is large and $pr_w[\bar{\chi}_5^2 \geq 5.049]$ is close to .05.

interested in comparing increasing dosages of a drug to a zero dose control. As Chase, and Williams (1971,1972) before him, noted, "researchers often increase the sample size on the zero dose control over the sample sizes of the nonzero dose levels." Assume that the control is indexed by 1. Chase developed an approximation for the case where $w_2 = w_3 = \cdots = w_k$ and $w_1/w_2 > 1$.

First, Chase found a recursion formula for the limiting values of $P_w(\ell,k)$; $\ell = 1,2,\cdots,k$ as $w_1 \to \infty$ while the common value of $w_2, w_3, \cdots, w_k$ is held fixed. We denote the limiting probabilities by $P_\infty(\ell,k)$. Chase tabled critical values for the $\bar{\chi}^2$ and $\bar{E}^2$ distributions associated with these limiting values. For values of (w_1/w_2) between 1 and ∞ he found that a linear interpolation in $(w_1/w_2)^{-1/2}$ between the equal weights case and this limiting case gives a very adequate approximation. For example, if C_∞ denotes the appropriate critical value computed from these limiting level probabilities and if C_1 denotes the appropriate equal weights critical value then the approximate critical value is given by $C = \{1-(w_1/w_2)^{-1/2}\}C_\infty + (w_1/w_2)^{-1/2}C_1$. Chase gives both exact $(k \le 4)$ and Monte Carlo $(k > 4)$ evaluations of his approximation and concludes that they provide quite satisfactory results. Our studies confirm this conclusion.

Starting with Chase's recursion formula, Robertson and Wright (1982b) derived the generating function of the sequence of limiting level probabilities, $k = 1,2,\cdots$. This generating function is given by

$$\Phi_{\infty,k}(s) = s\binom{1/2\ s + k - 3/2}{k-1}$$

for $k = 1,2,\cdots$ and, in turn, yields a sharper recursion formula for the limiting level probabilities, namely,

$$P_\infty(1,k+1) = (2k-1)P_\infty(1,k)/(2k) = (2k)!/(2^{2k}(k!)^2)$$

$$P_\infty(k+1,k+1) = P_\infty(k,k)/(2k) = (1/2)^k/k!$$

$$P_\infty(\ell,k+1) = P_\infty(\ell-1,k)/(2k) + (2k-1)P_\infty(\ell,k)/(2k).$$

Using this relation, it is a fairly simple matter to write a computer program to generate $P_\infty(\ell,k)$ for k up to any reasonable value such as 30. For $k \le 12$, these probabilities are tabled in Robertson and Wright (1982b). It was also noted in that work that the limiting values of $P_w(\ell,k)$ for the situation in which $w_1 = w_2 = \cdots = w_{k-1}$ and $w_k/w_1 \to \infty$ are also given by $P_\infty(\ell,k)$.

We develop an approximation for $P_w(\ell,k)$ based upon a pattern of large and small weights in w. We first obtain the limiting values for the $P_w(\ell,k)$ for each situation in which the w_i assume one of two values, one of which remains constant and the other approaches infinity. The approximation is an interpolation between the appropriate limiting case and the equal weights case. We try to convey the basic ideas behind the limiting cases by heuristically considering some examples.

Suppose $k = 3$ and that $w_1 = w_3 = \infty$ while $w_2 = 1$. Let $\mu_1^* \le \mu_2^* \le \mu_3^*$ be the maximum likelihood estimates of μ_1, μ_2, μ_3 subject to $\mu_1 \le \mu_2 \le \mu_3$ and recall that $P_w(\ell,3)$ is computed under the assumption that $\mu_1 = \mu_2 = \mu_3$. We may assume that their common value is zero without loss of generality. Recall, also, the pool adjacent violators algorithm for computing μ_1^*, μ_2^* and μ_3^*. This algorithm is discussed beginning on page 13 of Barlow et al. (1972). The value of w_i is proportional to the reciprocal of the variance of $\bar{x}_i$. Thus $\bar{x}_1$ and $\bar{x}_3$ are degenerate at 0 and $\bar{x}_2$ is a

unit normal variable. Either $\bar{x}_2 > 0$ in which case $\bar{x}_2$ forms a violation with $\bar{x}_3$ or $\bar{x}_2 < 0$ and forms a violation with $\bar{x}_1$. If $\bar{x}_2 < 0$ then it is amalgamated with $\bar{x}_1$ and they are both replaced by their weighted average and assigned weight w_1+w_2, which behaves like w_1. Since the weight on $\bar{x}_1$ is infinite the weighted average behaves like $\bar{x}_1$. Similarly, if $\bar{x}_2 > 0$, it and $\bar{x}_3$ are replaced by their weighted average, or $\bar{x}_3$ and assigned weight w_2+w_3, or w_3. Thus, $P_w(3,3) = 0$, $P_w(1,3) = P_{(w_1,w_3)}(1,2)$ and $P_w(2,3) = P_{(w_1,w_2)}(2,2)$. Since $P_w(1,2) = P_w(2,2)$ for any $0 < w_1, w_2 < \infty$, the limiting values are both $1/2$. Noting that $P_{(a,b,a)}(\ell,3) = P_{(1,b/a,1)}(\ell,3)$, we see that this case was treated rigorously by Robertson and Wright (1982a).

Now suppose $w_1 = w_3 = 1$ and $w_2 = \infty$. Then, $\bar{x}_2$ is degenerate at 0 while $\bar{x}_1$ and $\bar{x}_3$ are unit normal variables. If $\bar{x}_1 < 0 < \bar{x}_3$ then we will have three levels. In all the other cases, either $\bar{x}_1$ or $\bar{x}_3$ is amalgamated with $\bar{x}_2$. When one of these amalgamations takes place the two variables are replaced by a variable which is degenerate at zero. It follows that the number of levels is one less than the sum of the number of levels among μ_1^* and μ_2^* and the number of levels among μ_2^* and μ_3^*. Moreover, these two numbers of levels are independent. Thus, the sequence $\{P_w(\ell,3)\}_{\ell=1}^{3}$ is related to the convolution of the sequence $\{P_\infty(\ell,2)\}_{\ell=1}^{2}$ with itself. Specifically, if $\{P_\infty(\ell,2)\}_{\ell=1}^{2} * \{P_\infty(\ell,2)\}_{\ell=1}^{2} = \{C_\ell\}_{\ell=2}^{4}$ then C_ℓ is the probability of $\ell-1$ levels, $\ell = 2,3,4$.

Now consider an arbitrary k and suppose $i_1 < i_2 < \cdots < i_j$ is an ordered subset of $\{1,2,\cdots,k\}$; $w_{i_1} = w_{i_2} = \cdots = w_{i_j} = 1$ while the other weights are infinite. Let $I = k-j$ and suppose A is the number of finite weights on the left and B is the number of finite weights on the right.

Specifically, $w_1 = w_2 = \cdots = w_A = 1$, $w_{A+1} = \infty$, $w_{k-B} = \infty$ and $w_{k-B+1} = \cdots = w_k = 1$. If one uses the pool adjacent violators algorithm to order the first $A+1$ estimates and the last $B+1$ estimates, then in the former case the last estimate will behave like $\bar{x}_{A+1}$ with infinite weight on it and in the latter case the first estimate will behave like $\bar{x}_{k-B}$ with infinite weight. Thus, the number of levels is two less than the number of levels among $\mu_1^*, \cdots, \mu_{A+1}^*$ plus the number of levels among $\mu_{A+1}^*, \cdots, \mu_{k-B}^*$ plus the number of levels among $\mu_{k-B}^*, \cdots, \mu_k^*$. Moreover, these three numbers of levels are independent. The probabilities for the number of levels among $\mu_1^*, \cdots, \mu_{A+1}^*$ and among $\mu_{k-B}^*, \cdots, \mu_k^*$ are given by Chase. If $w_i = 1$ for any i such that $A+1 \le i \le k-B$ then the corresponding $\bar{x}_i$ must be amalgamated with one of the degenerate variables to one side of $\bar{x}_i$. Thus the probabilities for the number of levels among $\mu_{A+1}^*, \cdots, \mu_{k-B}^*$ are simply the equal weights probabilities, $\{P(\ell, I)\}_{\ell=1}^{I}$. Thus the sequence of level probabilities is related to a convolution of three sequences, each of which is easily computed. This idea is most succinctly expressed in terms of the generating function of the sequence of limiting level probabilities which we denote by $\Theta_{\infty,k}(s)$.

$$\Theta_{\infty,k}(s) = s^{-2}\, \Phi_{\infty,A+1}(s)\, \Phi_I(s)\, \Phi_{\infty,B+1}(s)$$

$$= s^{-2} s \binom{1/2\, s + A - 1/2}{A} \binom{s + I - 1}{I} s \binom{1/2\, s + B - 1/2}{B}$$

$$= \binom{1/2\, s + A - 1/2}{A} \binom{s + I - 1}{I} \binom{1/2\, s + B - 1/2}{B}.$$

A rigorous proof could be given but it would provide no new insights. The

$P(\ell, I)$ are tabled in Barlow et al. (1972) and the $P_\infty(\ell, A+1)$ are tabled in Robertson and Wright (1982b).

We recommend an approximation which is an interpolation between critical values (or P-values) computed from $\bar{\chi}^2$ or $\bar{E}^2$ distributions associated with the above limiting coefficients and distributions associated with the equal weights case. In order to recommend a specific approximation we need to make two decisions. The first decision is, for a given set of weights how to decide which are large and which are small. The second is how to interpolate. The criteria for choosing large and small weights and the method of interpolation seem to be interrelated and we tried a very large number of combinations, evaluating these combinations on the basis of randomly generated weight sets as in our evaluation of the equal weights approximation.

We tried calling weights large if they exceeded some fraction of the largest weight; if they exceeded some fraction of the sum of weights; or if they exceeded some value midway between the largest and smallest weights. We tried interpolating on various powers of the ratio of the largest weight to the smallest weight and the ratio of the average of those weights we called large and the average of the ones we called small. Choices between combinations were not always clear but almost everything we tried seemed to be preferable to an approximation based upon the equal weights case alone.

Based upon these studies, our recommendation is to call weights large if they exceed

$$.65 \min(w_1, w_2, \cdots, w_k) + .35 \max(w_1, w_2, \cdots, w_k)$$

and to interpolate on the basis of the ratio of the 1/3 power of the average of the large weights to the average of the small weights.

For example, suppose $k=5$, $w = (1,1.2,5.6,1.4,4.7)$ and we are interested in a 5 percent critical value for the $\bar{\chi}^2$ distribution associated with these weights. Then $.65 \min(w_1,w_2,\cdots,w_5) + .35 \max(w_1,w_2,\cdots,w_5) = .65 \cdot 1 + .35 \cdot 5.6 = 2.61$. Thus, call the first, second and fourth weights small and the third and fifth large. The values of A, B and I are given by $A = 2$, $B = 0$, $I = 2$. The generating function of our limiting coefficients is given by

$$\Theta_{\infty,5}(s) = \binom{1/2\, s + 3/2}{2}\binom{s+1}{2}$$

$$= \frac{1}{4}\left(\frac{s+3}{2}\right)\left(\frac{s+1}{2}\right) s(s+1)$$

$$= \frac{3}{16}s + \frac{7}{16}s^2 + \frac{5}{16}s^3 + \frac{1}{16}s^4 .$$

Thus, our limiting coefficients are $P_\infty(1,5) = \frac{3}{16}$, $P_\infty(2,5) = \frac{7}{16}$, $P_\infty(3,5) = \frac{5}{16}$, $P_\alpha(4,5) = \frac{1}{16}$, $P_\infty(5,5) = 0$. The five percent critical value of the associated $\bar{\chi}^2$ distribution is 4.8969. The five percent critical value of the equal weights $\bar{\chi}^2_5$ distribution is 5.0491. The ratio of the average large weight to the average small weight is 4.29. Thus, our approximate 5 percent critical value is

$$[1 - (4.29)^{-1/3}] \cdot 4.8969 + (4.29)^{-1/3} \cdot 5.0491 = 4.9906.$$

The actual five percent critical value is 5.0996 and the actual $\bar{\chi}^2_5$ distribution associated with these weights has the probability, .0527, to

the right of 4.9906. In other words, if we wanted a five percent test and used 4.9906 as our critical value we would actually have $\alpha = .0527$. This approximation also seems to give adequate approximations for the values of $P_w(\ell,k)$ and are thus adequate for computing P-values. In this example the actual values of $P_w(1,5),\cdots,P_w(5,5)$ are .1908, .4132, .3008, .0868, .0084. The approximation which places weight $(4.29)^{-1/3}$ on the equal weights values and $1 - 4.29^{-1/3}$ on the limiting values yields the values .1952, .4247, .2997, .0753, .0051.

We evaluate our approximation as follows. For an arbitrary set of 5 weights we use our procedure to compute an approximate 5 percent critical value, $a_{.05}(w)$. Since k is 5 we can, as before, compute an exact value for the probability, $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$. If our approximation is good, this should be close to .05. We did this for weights of the form $w = (1,\epsilon,\epsilon,\epsilon,1)$ and $w = (\epsilon,\sqrt{\epsilon},1,\sqrt{\epsilon},\epsilon)$ with $\epsilon = R^{-1}$ and R = 1,2,3,4,5,10,15,20,25,30. The results are given in Table 7. The numbers in parentheses are the values we obtained using the equal weights approximation. It should be noted that the approximation based upon the pattern of large and small weights performs exceptionally well for U-shaped weight sets.

In Table 8 we have reported a frequency distribution of $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ for 10,000 randomly generated weight sets. All 10,000 probabilities fell between .0314 and .0745 and 9,754 fell between .040 and .060. Recall that for the equal weights approximation all 10,000 fall between .0219 and .0803 while 7,353 of these probabilities fall between .040 and .060. The results of a study where the weight sets were randomly generated so as to control the ratio of the largest weight to the smallest weight are given

ϵ	R	$w = (1,\epsilon,\epsilon,\epsilon,1)$ $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$	$w = (\epsilon,\sqrt{\epsilon},1,\sqrt{\epsilon},\epsilon)$ $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$
1.0000	1	.0500 (.0500)	.0500 (.0500)
.5000	2	.0535 (.0420)	.0514 (.0557)
.3333	3	.0540 (.0375)	.0522 (.0590)
.2500	4	.0539 (.0346)	.0576 (.0612)
.2000	5	.0536 (.0324)	.0588 (.0629)
.1000	10	.0520 (.0267)	.0622 (.0679)
.0667	15	.0509 (.0240)	.0640 (.0705)
.0500	20	.0501 (.0224)	.0652 (.0723)
.0400	25	.0496 (.0213)	.0660 (.0736)
.0333	30	.0492 (.0205)	.0667 (.0746)

Table 7. $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ for various weight configurations.

in Table 9. It is interesting to note that for the 6000 weight sets generated so that $R \leq 10$, the approximation gave true α levels between .04 and .06.

In Table 8 note that every ratio which is associated with a probability outside of [.048,.052] is at least 1.5. This is only slightly larger than the value we obtained for the equal weights approximation. However, every ratio which is associated with a probability outside of the interval [.04,.06] is at least 4.7. The corresponding value for the equal weights approximation was 3.4. In the case $k = 5$ we conclude that if the ratio of the largest weight to the smallest weight is no more than 4.7 then this approximation should provide an adequate result.

In order to evaluate our approximation for larger k, we again considered randomly generated weight sets. With $k = 10$, weight sets of the form $w_i = u_i/\sum_{j=1}^{10} u_j$, $i = 1,2,\cdots,k$, were generated and for each weight set $a_{.05}(w)$, the approximate $\alpha = .05$ critical value was obtained. Of course, for $k = 10$, the true value of $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ cannot be calculated for arbitrary weight sets. Hence, it was estimated by Monte Carlo techniques based on 4,000 iterations. So, ten uniform variables were needed for each weight set as well as 40,000 normal variables to estimate the appropriate probability. This process was repeated for 1000 weight sets. The number of weight sets was limited by the amount of computer time required for these results. As can be seen, this project required the generation of over 40 million random variables. The frequency distribution for the estimated probabilities is given in Table 10. If one considers the approximation adequate if the estimated value of $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ is in [.04,.06], then it is interesting to note that 97.7% of the estimated values were in this range. This is a slight increase over the case $k = 5$;

Interval	Frequency	Minimum Ratio	Maximum Ratio	Interval	Frequency	Minimum Ratio	Maximum Ratio
[.012,.016]	0			[.056,.060]	492	3.3	2117.6
[.016,.020]	0			[.060,.064]	109	5.4	3101.4
[.020,.024]	0			[.064,.068]	28	21.2	9545.8
[.024,.028]	0			[.068,.072]	3	122.5	496.4
[.028,.032]	1	79.3	79.3	[.072,.076]	1	1238.1	1238.1
[.032,.036]	18	18.9	∞	[.076,.080]	0		
[.036,.040]	86	4.7	1414.1	[.080,.084]	0		
[.040,.044]	406	3.5	1026.5	[.084,.088]	0		
[.044,.048]	1730	1.5	∞	[.088,.092]	0		
[.048,.052]	4577	1.1	∞	[.092,.096]	0		
[.052,.056]	2549	1.5	8747.9				

Table 8. Frequency distribution for $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ with 10,000 randomly generated weight sets and $a_{.05}(w)$ computed using the approximation.

[a,b]	All 1000 values of $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ in:
[1/9,10/9]	[.0396,.0600]
[1/4,5/4]	[.0425,.0569]
[1/3,4/3]	[.0431,.0561]
[1/2,3/2]	[.0443,.0548]
[1,2]	[.0463,.0529]
[2,3]	[.0478,.0520]

Table 9. Bounds on $pr_w[\bar{\chi}_5^2 \geq a_{.05}(w)]$ when the weights are generated so that the ratio of the largest weight to the smallest weight is controlled.

and is an indication that the quality of the approximation is maintained as k is increased to $k = 10$.

For further evaluation, we considered data sets in which the values could be sample sizes for testing the homogeneity of a set of means versus a trend alternative. Under the assumption of equality of variances these values become the weights. We found approximate values for $P_w(\ell,k)$ by Monte Carlo techniques based on 4,000 iterations and compared these values to those obtained by our approximation.

We also compared the tail probabilities of the two $\bar{\chi}^2$ distributions corresponding to these two sets of approximate level probabilities. For example, in Table 24.2 on page 617 of Neter, Wasserman and Whitmore (1978) we found the number of twin-engine executive jets sold annually between 1968 and 1977 by an aircraft manufacturer. Taking these 10 numbers, 147, 175, 150, 191, 188, 179, 200, 220, 208, 330, as our weights we computed the values of our approximate $P_w(\ell,k)$'s which we shall denote by $P_{a,w}(\ell,10)$; $\ell = 1,2,\cdots,10$. The values of $P_{a,w}(\ell,10)$ and the relative frequencies of these counts which we denote by $P_{m,w}(\ell,k)$ are reported in Table 11. The agreement is good. The maximum absolute difference of $|P_{a,w}(\ell,10) - P_{m,w}(\ell,10)|$, $\ell = 1,2,\cdots,10$ is .0153 and occurs when $\ell =$ [illegible].

We also computed the tail probabilities of the $\bar{\chi}^2$ distributions corresponding to the two sets of coefficients. Specifically, we computed $P(\bar{\chi}^2_{10} \geq t)$ for $t = .5, 1.0, 1.5, \cdots, 30.0$. The maximum difference between these 60 tail probabilities was .00951, occurring at $t = 1.5$. The values of these tail probabilities at $t = 1,2,3,\cdots,15$ are given in Table 12. The magnitude of the difference for the larger values of t was on the order of 10%. For a significance level around .1 the difference was about .005,

Interval	Frequency	Interval	Frequency
[.032,.036]	0	[.052,.056]	204
[.036,.040]	7	[.056,.060]	75
[.040,.044]	68	[.060,.064]	14
[.044,.048]	265	[.064,.068]	2
[.048,.052]	365	[.068,.072]	0

Table 10. Frequency distribution of estimated values of $pr_w[\bar{\chi}^2_{10} \geq a_{.05}(w)]$ with 1000 randomly generated weight sets and $a_{.05}(w)$ computed using the approximation.

ℓ	$P_{m,w}(\ell,10)$	$P_{a,w}(\ell,10)$
1	.09425	.09555
2	.28975	.27445
3	.32600	.32092
4	.19850	.20498
5	.07300	.08018
6	.01600	.02021
7	.00225	.00333
8	.00025	.00035
9	.00000	.00002
10	.00000	.00000

Table 11. Comparison of $P_{a,w}(\ell,10)$ with $P_{m,w}(\ell,10)$ for $w = (147, 175, 150, 191, 188, 179, 200, 220, 298, 330)$.

t	$\sum_{\ell=1}^{10} P_{m,w}(\ell,10)P[\chi^2_{\ell-1} \geq t]$	$\sum_{\ell=1}^{10} P_{A,w}(\ell,10)P[\chi^2_{\ell-1} \geq t]$	Difference
1	.53300	.54302	.00903
[illegible]	.34873	.35814	.00941
[illegible]	.22856	.23663	.00807
[illegible]	.14935	.15572	.00637
[illegible]	.09722	.10201	.00479
[illegible]	.06306	.06655	.00349
[illegible]	.04076	.04325	.00249
[illegible]	.02627	.02801	.00174
[illegible]	.01688	.01809	.00121
[illegible]	.01082	.01165	.00083
11	.00692	.00748	.00056
12	.00442	.00479	.00037
13	.00281	.00307	.00026
14	.00179	.00196	.00017
15	.00114	.00125	.00011

Table 12. Comparison of the $\bar{\chi}^2$ distributions corresponding to Monte Carlo and approximate values of $P_w(\ell,10)$. $w = (147, 175, 150, 191, 188, 179, 200, 220, 208, 330)$.

for a significance level around .05 the difference was about .003, and for a significance level around .01 the difference was about .0008.

The equal weights $P(\ell,10)$ are given in Table A.5 of Barlow, et al. (1972). They also fit the $P_{m,w}(\ell,10)$ very well in this example and, in fact, they fit better than the approximation presented here. However, both are clearly adequate and, in general, we have seen that the approximation presented here outperforms the equal weights approximation.

We tried a number of weight sets with values of k larger than 5, obtaining similar results each time. For example, we took the weights to be the 12 values in Table 20.4 of Neter et al. (1978). The 12 weights together with the approximate and Monte Carlo values are given in Table 13. We also computed the tail probabilities of the two corresponding $\bar{\chi}^2$ distributions at $t = .5, 1.0, \cdots, 45$. The maximum absolute difference in these two functions was .002. The largest difference in these $\bar{\chi}^2$ tail probabilities corresponding to Monte Carlo P-values smaller than .15 was .00001. Comparing with the $P(\ell,12)$ from Table A.5 of Barlow et al. (1972), we see that the $P_{a,w}(\ell,12)$ fit better in this example. In fact the fit for tail probabilities less than or equal to .15 is remarkable.

Another weight set and corresponding $P_{a,w}(\ell,k)$ are presented in Table 14. The weights are taken from page 513 of Neter et al. (1978). The maximum absolute difference in the two $\bar{\chi}^2$ distributions corresponding to Table 14 is .01025 and the maximum absolute difference corresponding to Monte Carlo P-values less than .15 is .00415.

Thus, we have further evidence that the quality of the approximation holds up for $k > 5$.

ℓ	$P_{m,w}(\ell,12)$	$P_{a,w}(\ell,12)$
1	.08600	.09011
2	.26525	.26344
3	.32425	.31680
4	.20425	.21131
5	.09125	.08831
6	.02250	.02462
7	.00575	.00472
8	.00075	.00063
9	.00000	.00006
10	.00000	.00000
11	.00000	.00000
12	.00000	.00000

Table 13. Comparison of $P_{a,w}(\ell,k)$ with Monte Carlo values of $P_w(\ell,k)$ for $k = 12$ and $w = (155, 178, 215, 93, 128, 114, 172, 158, 197, 207, 95, 183)$.

ℓ	$P_{m,w}(\ell,15)$	$P_{a,w}(\ell,15)$
1	.07175	.06388
2	.21800	.21254
3	.30000	.30041
4	23950	.24174
5	.11900	.12459
6	.03875	.04375
7	.01175	.01085
8	.00125	.00195
9	.00000	.00026
10	.00000	.00002
11	.00000	.00000
12	.00000	.00000
13	.00000	.00000
14	.00000	.00000
15	.00000	.00000

Table 14. Comparison of $P_{a,w}(\ell,k)$ with Monte Carlo values of $P_w(\ell,k)$ for $k = 15$ and $w = (55, 20, 35, 45, 40, 25, 55, 30, 60, 45, 35, 25, 45, 35, 30)$.

4. Siskind's approximation. Siskind (1976) developed a very good approximation for $P_w(\ell,k)$ based upon a Taylor's expansion. The level probabilities depend upon the weights only through a matrix, R, of partial correlations. Let Q be the correlation matrix corresponding to the equal weights case. Thinking of the level probabilities as functions of this matrix, Siskind's approximation was based upon Taylor's expansions of these level probabilities about the equal weights matrix, Q. His approximation requires the values of certain derivatives of the $P(\ell,k)$'s evaluated at Q. Siskind provides a table of these values for $k \leq 8$. Thus, his technique is only usable for $k \leq 8$.

Let $s_{.05}(w)$ denote the five percent critical value computed from Siskind's approximation for the weight set $w = (w_1,w_2,w_3,w_4,w_5)$ of size 5. In Table 15 we have a frequency distribution for $pr_w[\bar{\chi}_5^2 \geq s_{.05}(w)]$ for 2,000 randomly generated weight sets. All 2000 probabilities fell between .0253 and .0515 and 1884 fell between .040 and .0515. The percentage which fell between .04 and .06 is slightly less than the percent that fell between these values for the approximation based upon the pattern of large and small weights. It seems unlikely that Siskind's approximation would ever give a value larger than .052 and this is a very distinct advantage for this approximation. In Siskind's study of his approximation, he found that it did not perform well for U-shaped weights. In those cases his approximation produced values that were considerably smaller than the true tail probabilities. Recall that in this case the approximation based upon the pattern of large and small weights performs quite well. Siskind's approximation seems to be more likely to give a very low value than the approximation based upon the pattern of large and small weights and the

Interval	Frequency	Minimum Ratio	Maximum Ratio	Interval	Frequency	Minimum Ratio	Maximum Ratio
[.012,.016]	0			[.056,.060]	0		
[.016,.020]	0			[.060,.064]	0		
[.020,.024]	0			[.064,.068]	0		
[.024,.028]	4	196.8	819.5	[.068,.072]	0		
[.028,.032]	14	70.5	2471.8	[.072,.076]	0		
[.032,.036]	27	27.9	3126.6	[.076,.080]	0		
[.036,.040]	71	20.7	1076.2	[.080,.084]	0		
[.040,.044]	179	7.1	3383.0	[.084,.088]	0		
[.044,.048]	481	2.9	1147.1	[.088,.092]	0		
[.048,.052]	1224	1.3	506.2	[.092,.096]	0		
[.052,.056]	0						

Table 15. Frequency distribution for $pr_w[\overline{\chi}_5^2 \geq s_{.05}(w)]$ with 2000 randomly generated weight sets. The value of $s_{.05}(w)$ is computed using Siskind's approximation.

fact that Siskind's approximation is only valid when $k \leq 8$ is a definite drawback.

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The use of much of the distribution theory developed for order restricted inference has been limited by the lack of computation algorithms for the "level probabilities" encountered in that theory. An approximation for these level probabilities, which accounts for the pattern of large and small "weights" is developed. This approximation and the equal weights approximation are examined in several different ways including the use of randomly generated weight sets. Both approximations appear

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to be reasonable for weight sets having a moderate amount of variability. The quality of the equal weights approximation, as a function of the amount of variability in the weights, deteriorates more quickly for certain patterns of large and small weights than for others. Thus, the approximation based upon the pattern of large and small weights is a significant improvement over the equal weights approximation. Finally, Siskind's (1976) approximation, which can be applied if the number of parameters is not too large, is discussed.